4. COMPLEX AMPLITUDE METHOD

4.1. Elements of Calculation of Harmonic Current Circuits by the Complex Amplitude Method

In a linear circuit with harmonic stimuli, the values of voltage and current can be found by directly solving the differential equations of the circuit. However, this task is very labour-intensive even in the case of a relatively simple circuit. That's why for calculation of harmonic current circuits is used the complex amplitude method the main point of which is using complex values for the currents and voltages acting in a circuit.

As a result, the operations of differentiation and integration are replaced by the operations of multiplication and division by the imaginary frequency $j\omega$ respectively, i.e. we move from differential to algebraic equations.

In addition, the use of complex values can simplify other operations on harmonic values. So, addition and subtraction of harmonic values are reduced to addition and subtraction of the real and imaginary parts of complex numbers represented in algebraic form. Multiplication and division of harmonic values are reduced to multiplication and division of the module and to addition and subtraction of the arguments of complex numbers represented in exponential form.

Thus, the general procedure of calculation of harmonic current circuits can be reduced to the following:

1. Moving from harmonic values (originals) to complex values (images) to get an equivalent complex circuit (ECC).
2. Calculating the obtained ECC by using known methods.
3. Moving from complex values (images) to real values (originals).

The complex amplitude method was developed by American scientists C. Steinmetz and A. Kennely in 1893–1894 and since then it has spread to calculate stationary harmonic current circuits.

4.2. Complex Resistance and Conductivity

Let the voltage in an electric circuit (Fig. 4.1) be

$$u(t) = U_m \cos(\omega t + \psi_u) \doteqdot \dot{U}_m = U_m e^{j\psi_u}.$$

The current flowing through it is

$$i(t) = I_m \cos(\omega t + \psi_i) \doteqdot \dot{I}_m = I_m e^{j\psi_i}.$$

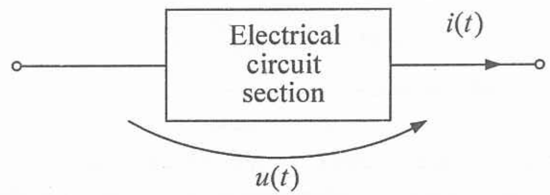


Fig. 4.1

The value

$$\dot{Z} = \frac{\dot{U}_m}{\dot{I}_m} = \frac{U_m e^{j\psi_u}}{I_m e^{j\psi_i}} = \frac{U_m}{I_m} e^{j(\psi_u - \psi_i)} = Z e^{j\varphi} = r + jx$$

is called the complex impedance of the circuit. Here

$$Z = Mod[\dot{Z}] = \sqrt{r^2 + x^2} \qquad (4.1)$$

is the total impedance,

$$\varphi = \mathrm{Arg}[\dot{Z}] = \psi_u - \psi_i = \mathrm{atan}\frac{x}{r} \qquad (4.2)$$

is the phase shift between the voltage and the current,

$$r = Z\cos\varphi \qquad (4.3)$$

is the resistance of the circuit,

$$x = Z\sin\varphi \qquad (4.4)$$

is the reactance of the circuit.

The complex impedance $\dot{Z}$, as follows from (4.1)–(4.4), can be represented on the complex plane (Fig. 4.2).

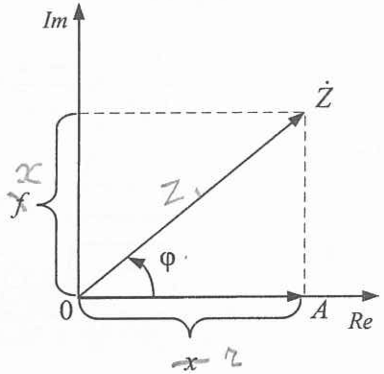


Fig. 4.2

Here the resistance r, reactance x and total impedance Z are convenient for representation on the complex plane of the impedances of complex circuit elements with different characters of these impedances and various connections between them.

The triangle OZA is called the impedance triangle. The value

$$\dot{Y} = \frac{\dot{I}_m}{\dot{U}_m} = \frac{I_m e^{j\psi_i}}{U_m e^{j\psi_u}} = \frac{I_m}{U_m} e^{-j(\psi_u - \psi_i)} = y e^{j\varphi} = g - jb$$

is called the complex admittance of the circuit. Here

$$y = Mod[\dot{Y}] = \sqrt{g^2 + b^2} \tag{4.5}$$

is called the total admittance,

$$g = y\cos\varphi \tag{4.6}$$

is the conductance of the circuit (real part of admittance),

$$b = y\sin\varphi \tag{4.7}$$

is the susceptance of the circuit (imaginary part of admittance).

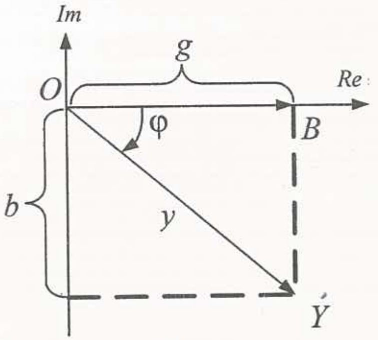


Fig. 4.3

The complex admittance Y, as follows from (4.5)–(4.7), can be represented on the complex plane (Fig. 4.3). Here the conductance g, susceptance b and complex admittance $\dot{Y}$ are shown as vectors.

The triangle OYB is called the admittance triangle. Since

$$\dot{Y} = \frac{1}{\dot{Z}}$$

then

$$g - jb = \frac{1}{r + jx} = \frac{r - jx}{r^2 + x^2} = \frac{r}{r^2 + x^2} - j\frac{x}{r^2 + x^2} = \frac{r}{z^2} - j\frac{x}{z^2},$$

i.e.

$$g = \frac{r}{z^2}; \quad b = \frac{x}{z^2}.$$

Also

$$r + jx = \frac{1}{g - jb} = \frac{g + jb}{g^2 + b^2} = \frac{g}{g^2 + b^2} + j\frac{b}{g^2 + b^2} = \frac{g}{y^2} + j\frac{b}{y^2},$$

i.e.

$$r = \frac{g}{y^2}; \quad x = \frac{b}{y^2}.$$

4.3. Harmonic Current Circuit Power

The instantaneous ~~capacity~~ power $p(t)$ of the electric circuit (see Fig. 4.1) is determined by the expression

$$\begin{aligned} p(t) &= i(t)u(t) = I_m\cos(\omega t + \psi_i)U_m\cos(\omega t + \psi_u) = \\ &= \sqrt{2}I\sqrt{2}U\frac{1}{2}\left[\cos(2\omega t + \psi_u + \psi_i) + \cos(\psi_u - \psi_i)\right] = \\ &= UI\cos\varphi + UI\cos(2\omega t + \psi_u + \psi_i). \end{aligned} \tag{4.8}$$

Fig. 4.4 shows the curves of current, voltage and power in the circuit.

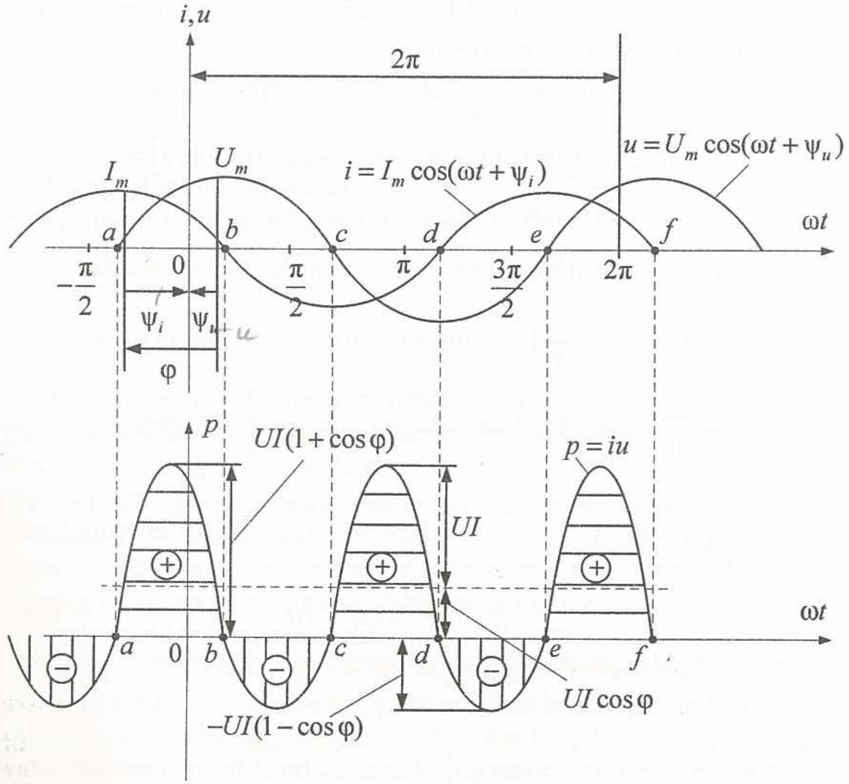


Fig. 4.4

Here, on the intervals *a-bc-de-f*, when the instantaneous voltage and current have the same sign, the instantaneous power is positive — the electric circuit consumes power. On the intervals *b-c-d-e*, when the instantaneous voltage and current have different signs, the instantaneous power is negative — the electric circuit supplies power to the other circuits.

The average power for a period T is determined by the integral

$$P=\frac{1}{T}\int_0^T p(t)\,dt=\frac{1}{T}\int_0^T\left[UI\cos\varphi+UI\cos(2\omega t+\psi_u+\psi_i)\right]dt=$$

$$=\frac{UI}{T}\left[\cos\varphi\int_0^T dt+\int_0^T\cos(2\omega t+\psi_u+\psi_i)\,dt\right]=UI\cos\varphi$$

and is called active power. Active power is measured in watts (W).

The quantity cos φ is called the power factor.

If the voltage or current in (4.8) is shifted by π/2, the instantaneous power is determined by the expression

$$p(t)=i(t)u(t)=I_m\cos(\omega t+\psi_i)U_m\sin(\omega t+\psi_u)=$$

$$=\sqrt{2}I\sqrt{2}U\frac{1}{2}\left[\sin(2\omega t+\psi_u+\psi_i)+\sin(\psi_u-\psi_i)\right]=$$

$$=UI\sin\varphi+UI\sin(2\omega t+\psi_u+\psi_i).$$

The average power for a period T is determined by the integral

$$Q=\frac{1}{T}\int_0^T p(t)\,dt=\frac{1}{T}\int_0^T\left[UI\sin\varphi+UI\sin(2\omega t+\psi_u+\psi_i)\right]dt=$$

$$=\frac{UI}{T}\left[\sin\varphi\int_0^T dt+\int_0^T\sin(2\omega t+\psi_u+\psi_i)\,dt\right]=UI\sin\varphi$$

and is called reactive power. Reactive power is measured in volt-amperes reactive (var).

The value

$$S=\sqrt{P^2+Q^2}=UI \tag{4.9}$$

is called the total power ~~capacity~~, and it is measured in volt-amperes (VA).

Complex power $\tilde{S}$ is defined as the product of the complex effective value of voltage $\dot{U}$ and the complex-conjugate effective value of current $\overset{*}{I}$. ~~From (3.14) and (3.15)~~ we get:

$$\tilde{S}=\dot{U}\overset{*}{I}=Ue^{j\psi_u}Ie^{-j\psi_i}=UIe^{j(\psi_u-j\psi_i)}=$$

$$=UIe^{j\varphi}=UI\cos\varphi+jUI\sin\varphi=P+jQ= \tag{4.10}$$

$$=Se^{j\varphi}=S\cos\varphi+jS\sin\varphi$$

i.e.

$$P=\mathrm{Re}\left[\tilde{S}\right]=S\cos\varphi=UI\cos\varphi, \tag{4.11}$$

$$Q=\mathrm{Im}\left[\tilde{S}\right]=S\sin\varphi=UI\sin\varphi, \tag{4.12}$$

$$S=\mathrm{Mod}\left[\tilde{S}\right]=UI, \tag{4.13}$$

$$\varphi=\mathrm{ARG}\left[\tilde{S}\right]=\mathrm{atan}\frac{Q}{P}. \tag{4.14}$$

Using (4.9)–(4.14) we can build a power triangle OSA on the complex plane (Fig. 4.5).

The power factor is

$$\cos\varphi=\frac{P}{S}.$$

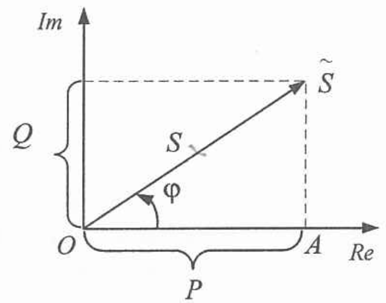


Fig. 4.5

The value of the power factor plays an extremely important role in electric power systems. It determines the amount of reactive power Q circulating in the system. It varies between the energy source and the consumer, twice during the period T coming to the consumer, being accumulated in the form of the energy of magnetic field inductance or the energy of electric field capacitance, and returning to the source. I.e. a reactive current flows through the circuit elements and causes additional losses. The more the power factor cos φ, the more active power P is supplied to the load for specified effective values of the current I and voltage U. Therefore, to increase cosφ in the case of active-reactive load, we must compensate the reactivity of the load by the opposite reactivity. As the impedance of the majority of real industrial systems is of active-inductive nature, batteries of capacitors are used for compensation. The capacity of a battery is chosen so that its reactance or conductance is equal to the reactance or conductance of the load respectively. A compensating battery is connected in series (series compensation) or in parallel with the load (transversal compensation). In most industrialized countries, the reactive power of compensating capacitors is close in magnitude to the active power of loads. For example, in the USA or Germany, 1 kW of active power accounts for 1 kWar of compensational reactive power. With such proportions the power of electrical energy sources is used in modes close to optimal ones.

4.4. Active Power Transfer from the Source to the Load

Consider the circuit shown in Fig. 4.6. Here, the source of voltage with an EMF $\dot{E}_m$ and internal impedance $Z_{in} = r_{in} + jx_{in}$ creates a current I_m through the load resistance $Z_l = r_l + jx_l$.

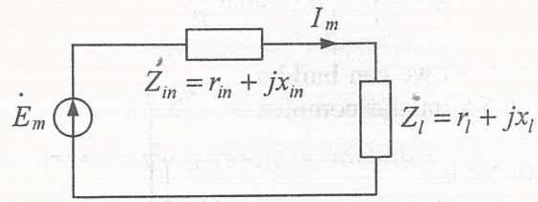


Fig. 4.6

The main criteria for matching the energy source with the load are:

1) transferring the maximum active energy to the load,
2) maximum efficiency.

In low-power information circuits it is necessary to get in the load the maximum power of the signal carrying useful information, regardless of efficiency. Let us consider this criterion.

The resultant complex impedance of the circuit is

$$\dot{Z} = r + jx = \dot{Z}_{in} + \dot{Z}_l = r_{in} + jx_{in} + r_l + jx_l = (r_{in} + r_l) + j(x_{in} + x_l).$$

The effective current of the circuit:

$$I = \frac{I_m}{\sqrt{2}} = \frac{E}{\sqrt{r^2 + x^2}} = \frac{E}{\sqrt{(r_{in} + r_l)^2 + (x_{in} + x_l)^2}}.$$

The active power of the load:

$$P_l = I^2 r_l = \frac{E^2 r_l}{(r_{in} + r_l)^2 + (x_{in} + x_l)^2}.$$

If

$$x = x_{in} + x_l = 0,$$

this power will be maximum, i.e. the condition should be observed:

$$x_{in} = -x_l. \tag{4.15}$$

Then we get the power:

$$P_{l_{\max}} = \frac{E^2 r_l}{(r_{in} + r_l)^2}. \tag{4.16}$$

Determine the relationship between r_{in} and r_l for which a maximum power $P_{l_{\max}}$ can be achieved. Let us take the derivative of $P_{l_{\max}}$ with respect to r_l

$$\frac{dP_{l_{\max}}}{dr_l} = \frac{E^2\left[(r_{in} + r_l)^2 - 2(r_{in} + r_l)\right]}{(r_{in} + r_l)^4} = 0.$$

Hence

$$x_{in} = x_l. \tag{4.17}$$

After substituting (4.17) into (4.16) we get

$$P_{l_{mm}} = \frac{E^2}{4r_l}. \tag{4.18}$$

Expression (4.18) represents the maximum power that the energy source can supply to the load. 17

Combining (4.15) and (4.23) we get the condition under which the maximum power is transferred from the energy source to the load:

$$\dot{Z}_l = r_l + jx_l = r_{in} - jx_{in} = \overset{*}{Z}_{in}. \tag{4.19}$$

I.e. the complex impedance of the load must be equal to the complex-conjugate internal impedance of the energy source.

In high-power energy-generating plants the key indicator of efficiency and cost effectiveness is η. From condition (4.19) we get

$$P_s = EI_{mm} = E\frac{E}{2r_l} = \frac{E^2}{2r_l},$$

$$\eta = \frac{P_{l_{mm}}}{P_s} = \frac{E^2 2r_l}{4r_l E^2} = 0{,}5.$$

That is, when transferring the maximum active power, the efficiency is 50 % only. Electric power systems cannot work with such a low efficiency. In the common case

$$\eta=\frac{P_l}{P_s}=\frac{I^2 r_l}{I^2\left(r_{in}+r_l\right)}=\frac{r_H}{r_{in}+r_l}. \tag{4.20}$$

Therefore from (4.20) the condition of maximum efficiency is

$$r_l >> r_{in.} \tag{4.21}$$

4.5. Power Balance in a Harmonic Current Circuit

According to the law of energy conservation the sum of the instantaneous powers of all elements of a circuit is equal to zero:

$$\sum_{k=1}^{n} P_k = 0. \tag{4.22}$$

Equation (4.22) is called the balance condition of instantaneous powers. Using complex representation we get

$$\sum_{k=1}^{k} \tilde{S}_k = \sum_{k=1}^{n} P_k + j\sum_{k=1}^{n} Q_k = 0. \tag{4.23}$$

From here

$$\sum_{k=1}^{n} P_k = 0, \quad \sum_{k=1}^{n} Q_k = 0. \tag{4.24}$$

That is the sum of the active powers as well as the sum of the reactive powers of all circuit elements is equal to zero.

As energy sources (active circuit elements) supply energy, and consumers (passive circuit elements) consume energy, it obviously follows from (4.24) that

$$\sum_{k=1}^{n} P_{ks} = \sum_{l=1}^{m} P_{lcon}; \qquad \sum_{k=1}^{n} Q_{ks} = \sum_{k=1}^{m} Q_{lcon},$$

where

$$P_{ks} = U_k I_k \cos\varphi_k; \ P_{lcon} = I_l^2 r_l;$$

$$Q_{ks} = U_k I_k \sin\varphi_k; \ Q_{lcon} = I_l^2 x_l,$$

r_l, x_l are the load resistance and reactance respectively.

4.6. ~~A~~ Harmonic Current Circuit with One Passive Element

For circuits in Fig. 4.7, *a* the following expressions can be written according to Ohm's law for the instantaneous values of current in and voltage across the resistance r

$$u = ir, \quad i=\frac{u}{r}, \quad i = ug, \quad u=\frac{i}{g}. \tag{4.25}$$

For expressions (4.25) the images can be written as complex amplitudes

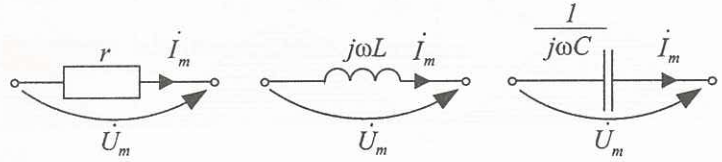
$$\dot{U}_m = U_m e^{j\psi_u} = r\dot{I}_m = Z_r \dot{I}_m = rI_m e^{j\psi_i} \tag{4.26}$$

or

$$U_m = rI_m, \ \psi_u = \psi_i, \ \varphi = \psi_u - \psi_i = 0. \tag{4.27}$$

That is, the current in and the voltage across the ~~active~~ resistance are in identical phases (coincide in phase). The complex impedance for the resistance is $Z_r = r$.

An equivalent complex circuit for the resistance (see Fig. 1.2, *a*) according to (4.26) and (4.27) is shown in Fig. 4.7, *a*.



a *b* *c*

Fig. 4.7

Similarly, from (4.27) we get

$$\dot{I}_m = \frac{\dot{U}_m}{r} = \frac{\dot{U}_m}{Z_r} = Y_r \dot{U}_m, \ Y_r = \frac{1}{r} = g,$$

which is the complex admittance of the resistance.

The power at the active resistance is purely active.

$$\tilde{S} = \dot{U}\overset{*}{I} = UIe^{j\varphi} = UI\cos\varphi + jUI\sin\varphi = UI\cos\varphi = P = UI.$$

The reactive power is:

$$Q = UI\sin\varphi = 0.$$

The complex currents and voltages as well as the above-mentioned impedances and admittances are represented on a complex plane as vectors, the lengths of which are proportional to the modulus of current or voltage, and the angle of rotation about the x-axis is proportional to their phases. A diagram representing a set of vectors on a complex plane is called a vector diagram. Time and vector diagrams for the resistance are given in Fig. 4.8 and Fig. 4.9 respectively.

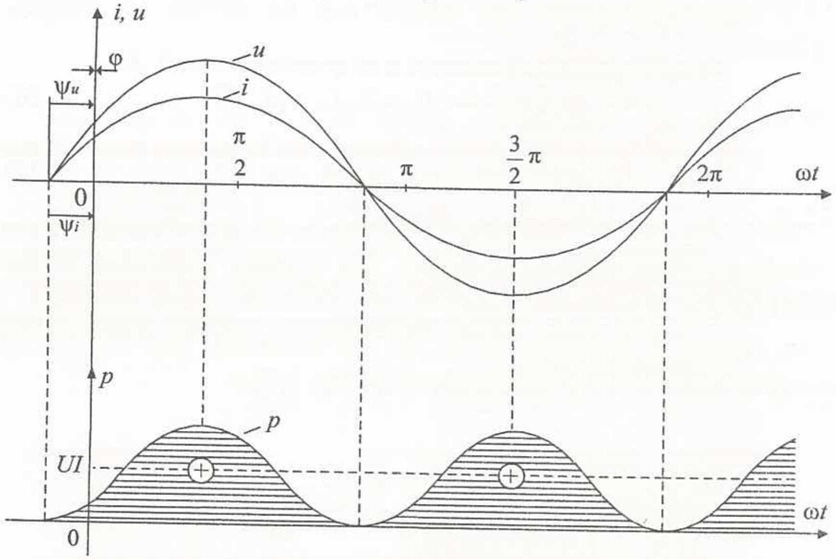


Fig. 4.8

For Fig. 4.7, b the following expressions can be written for the instantaneous values of current in and voltage across the inductance L

$$U = L\frac{di}{dt}, \quad i = \frac{1}{L}\int_0^t u\,d\tau.$$

For expression (~~4.15~~ 3.12) the images can be written as complex amplitudes

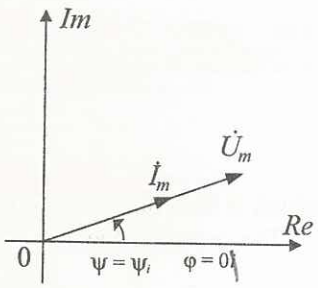


Fig. 4.9

$$\dot{U}_m = U_m e^{j\psi_u} = j\omega L \dot{I}_m = z_L \dot{I}_m = \omega L I_m e^{j(\psi_i + \frac{\pi}{2})} \qquad (4.28)$$

or

$$U_m = \omega L I_m; \quad \psi_u = \psi_i + \frac{\pi}{2}; \quad \varphi = \psi_u - \psi_i = \frac{\pi}{2}. \qquad (4.29)$$

That is, the voltage across the inductance is ahead of the current in ~~it~~ опережением ~~by an~~ angle of $\frac{\pi}{2}$. The complex impedance for the inductance L is

$$Z_L = j\omega L = jx_L,$$

where,

$$x_L = \omega L$$

is the coil's inductive impedance.

An equivalent complex circuit for the inductance (~~see Fig. 4.7, b~~) according to (4.28) and (4.29) is given in Fig. 4.7, b.

Similarly, from (4.2~~9~~8) we get

$$\dot{I}_m = \frac{\dot{U}_m}{j\omega L} = \frac{\dot{U}_m}{Z_L} = Y_L \dot{U}_m,$$

$$Y_L = \frac{1}{Z_L} = \frac{1}{j\omega L} = \frac{1}{jx_L} = -j\frac{1}{x_L} = -jb_L$$

is the complex admittance of the inductance.

$$b_L = \frac{1}{x_L} = \frac{1}{\omega L}$$

is the coil's inductive conductance.

The power at the inductance is positive and purely reactive.

$$\tilde{S} = \dot{U}\overset{*}{I} = UIe^{j\varphi} = UIe^{j\frac{\pi}{2}} = UI\cos\varphi + jUI\sin\varphi = jUI\sin\frac{\pi}{2} = jQ.$$

The active power:

$$P = UI\cos\varphi = UI\cos\frac{\pi}{2} = 0.$$

The reactive energy accumulated in the inductance according to (1.11) is

$$w_L = \frac{Li^2}{2} = \frac{L}{2}I_m^2\cos^2(\omega t + \psi_i) = \frac{LI_m^2}{4}\left[1 + \cos 2(\omega t + \psi_i)\right].$$

The maximum energy:

$$W_{L\max} = \frac{LI_m^2}{2} = LI^2.$$

Time and vector diagrams for the inductance are shown in Fig. 4.10 and Fig. 4.11 respectively.

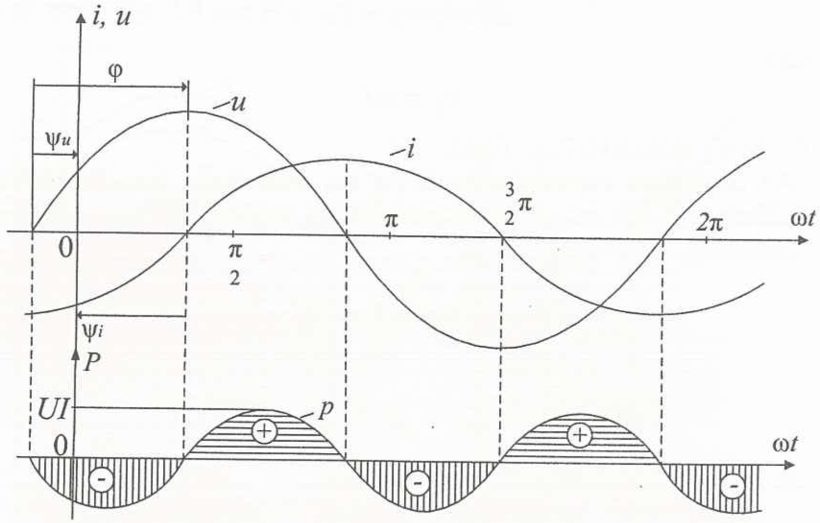


Fig. 4.10

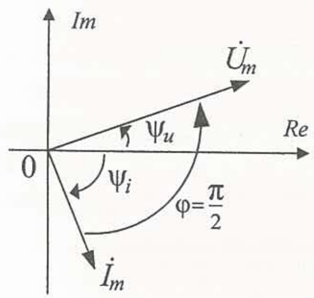


Fig. 4.11

The following expressions can be written for the instantaneous values of current ~~in~~ and voltage across the capacitance in Fig. 4.7, *c*

$$i = C\frac{du}{dt}; \qquad u = \frac{1}{C}\int_0^t i d\tau. \tag{4.30}$$

For expressions (4.30) the images can be written as complex amplitudes

$$\dot I_m = I_m e^{j\psi_i} = j\omega C \dot U_m = \overset{\omega C}{Y_C} \dot U_m e^{j\left(\psi_u + \frac{\pi}{2}\right)} \tag{4.31}$$

or

$$I_m = \omega C U_m; \quad \psi_i = \psi_u + \frac{\pi}{2}; \quad \varphi = \psi_u - \psi_i = -\frac{\pi}{2}. \tag{4.32}$$

It means that the voltage across the capacitance lags behind the current in it by an angle of $\frac{\pi}{2}$. The complex admittance for the capacitance C is:

$$Y_C = j\omega C = jb_C,$$

where

$$b_C = \omega C$$

is the capacitive conductance of the capacitor.

An equivalent complex circuit for the capacitance according to (4.31) and (4.32) is given in Fig. 4.7, *c*.

Similarly, from (4.30)

$$\dot U_m = \frac{\dot I_m}{j\omega C} = Z_C \dot I_m,$$

where

$$Z_C = \frac{1}{Y_C} = \frac{1}{j\omega C} = \frac{1}{jb_C} = -j\frac{1}{b_C} = -jx_C \tag{4.33}$$

is the complex impedance of the capacitance.

$$x_C = \frac{1}{b_C} = \frac{1}{\omega C}$$

is the capacitive reactance (impedance) of the capacitor.

The power at the capacitance is negative and purely reactive:

$$\tilde S = \dot U \overset{*}{I} = UIe^{j\varphi} = UIe^{-j\frac{\pi}{2}} =$$

$$= UI\cos\varphi - jUI\sin\varphi = -jUI\sin\frac{\pi}{2} = -jQ.$$

The active [true] power:

$$P = UI\cos\varphi = UI\cos\left(-\frac{\pi}{2}\right) = 0.$$

The reactive energy accumulated in the capacitance (1.13) is

$$w_C = \frac{Cu^2}{2} = \frac{C}{2} U_m^{\ 2} \cos^2(\omega t + \psi_u) = \frac{CU_m^{\ 2}}{4}\left[1 + \cos 2(\omega t + \psi_u)\right].$$

The maximal energy:

$$W_{C_{\max}} = \frac{CU_m^{\ 2}}{2} = CU^2.$$

Time and vector diagrams for the capacitance are shown in Fig. 4.12 and Fig. 4.13 respectively.

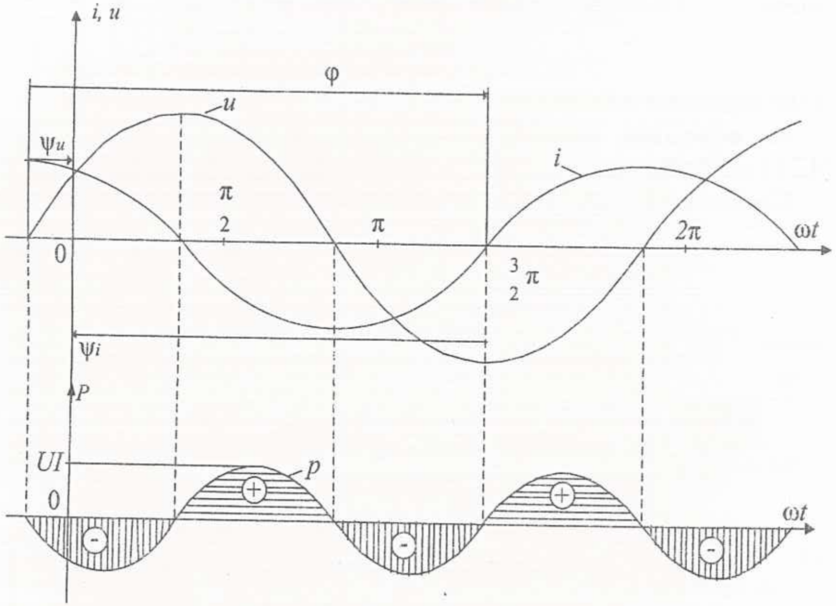


Fig. 4.12

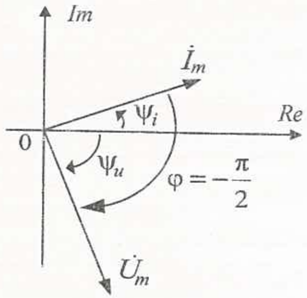
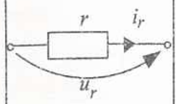
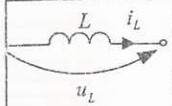
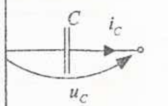


Fig. 4.13

The basic relationships for a harmonic current circuit with one passive element are presented in Table 4.1.

Table 4.1

Circuit elements	Relationships between current and voltage		Impedance	Power
	For instantaneous values	For complex amplitudes		
	$u = ri$	$\dot{U}_m = r\dot{I}_m$	$Z_r = r$	$P = I^2 r$
	$u = L\frac{di}{dt}$	$\dot{U}_m = j\omega L\dot{I}_m$	$Z_L = jx =$ $jx_L = j\omega L$ $x = x_L = \omega L$	$Q_L = I^2 x_L$
	$u = \frac{1}{C}\int_0^t i d\tau$	$\dot{U}_m = \frac{1}{j\omega C}\dot{I}_m$	$Z_C = jx =$ $jx_C = \frac{1}{j\omega C}$ $x = x_C = -\frac{1}{\omega C}$	$Q_C = -I^2 x_C$